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# DYNAMICS AND STABILITY OF ELASTIC COSSERAT CURVES†

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**Abstract**—A nonlinear dynamical theory of plane motions of a class of Cosserat curves is obtained and shown to include a classical type elastica theory as a special case. The undeformed state of a simply-supported curve is proved stable with respect to suitable metrics, provided the strain energy function is positive definite.

## 1. INTRODUCTION

NONLINEAR deformation theories of rods can be formulated by regarding a rod as a one-dimensional directed continuum, i.e. a curve with a triad of directors or deformable vectors defined at each point of the curve. The deformation of this "directed" curve consists of displacements of the points on the curve and independent stretches and rotations of the directors. The directors can be interpreted as material elements in the cross section of a rod and account for shearing, bending and twisting effects. The definition of a rod as a curve with a triad of directors leads to a complete description of the strain in a rod, as shown by Ericksen and Truesdell [1].

A nonlinear dynamical theory of elastic directed curves was developed by Whitman and DeSilva [2], who postulated a Hamilton's principle, conservation of mass and invariance of the action density function under rigid body variations as governing the dynamical behavior of elastic directed curves. These postulates were shown to yield a complete dynamical theory, i.e. equations of motion, boundary conditions and nonlinear constitutive equations. As a special case, the directors were constrained to be a rigid triad and a general theory of Cosserat curves was obtained. The general theory of Ref. [2] can be considered as a generalization of the statical theory of elastic directed curves presented by Cohen [3].

In this paper we further investigate the dynamics of Cosserat curves by restricting the curve to plane motions. In Section 2 a consistent nonlinear theory is obtained and shown to reduce to a classical type elastica theory by imposing the additional constraint that the director frame rotate with the unit tangent vector to the curve. We then pass to a linear theory and obtain a set of three linear displacement equations of motion. In Section 3 a definition of dynamical stability and a stability theorem due to Movchan [4] are stated. We then construct a suitable energy functional which is valid for a stability investigation of any general motion of an elastic directed curve. In Section 4 this functional is reduced to the case of plane motions of a simply-supported Cosserat curve, and Movchan's theorem leads to the result that the undeformed state of the curve is stable provided the strain energy function is positive definite.

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## 2. DYNAMICS OF PLANE COSSERAT CURVES

We consider here Cosserat curves, i.e. curves with rigid directors and restrict ourselves to the case when these rigid directors are a unit orthogonal triad. The general equations for this class of elastic Cosserat curves were derived by Whitman and DeSilva [2] and are given by

$$
\begin{aligned}
\frac{\partial \tau_\alpha}{\partial s} + w_{\alpha\beta}\tau_\beta + \rho f_\alpha &= \rho \dot{v}_\alpha + \rho e_{\alpha\beta\delta}\Omega_\beta v_\delta \\
\frac{\partial m_\alpha}{\partial s} + w_{\alpha\beta}m_\beta - e_{\alpha\beta\delta}\tau_\beta t_\delta + \rho l_\alpha &= \rho \dot{\alpha}_\alpha + \rho e_{\alpha\beta\delta}\Omega_\beta \alpha_\delta
\end{aligned} \tag{2.1}
$$

$$
\tau_\alpha = \rho\lambda \frac{\partial \varepsilon}{\partial y_\alpha}, \qquad m_\alpha = \rho\lambda e_{\alpha\beta\delta}\frac{\partial \varepsilon}{\partial F_{\delta\beta}} \tag{2.2}
$$

$$
y_\alpha = \lambda t_\alpha = \mathbf{d}_\alpha . \hat{\mathbf{r}}, \qquad F_{\alpha\beta} = \lambda w_{\alpha\beta} = \mathbf{d}_\alpha . \hat{\mathbf{d}}_\beta \tag{2.3}
$$

$$
\alpha_\alpha = B_{\alpha\beta}\Omega_\beta, \qquad B_{\alpha\beta} = A_{\theta\theta}\delta_{\alpha\beta} - A_{\alpha\beta} = B_{\beta\alpha}. \tag{2.4}
$$

In these equations $\rho$ is the mass density per unit length of the deformed curve $c$, $\boldsymbol{\tau}$ is the stress vector, $\mathbf{m}$ the couple stress vector, $\mathbf{f}$ the body force and $\mathbf{l}$ the body couple. The vector $\mathbf{v}$ is the velocity of points on $c$, and $\boldsymbol{\Omega}$ is the spin velocity of the rigid director triad $\mathbf{d}_\alpha$. The function $\varepsilon$ is the strain energy density, $y_\alpha$ and $F_{\alpha\beta}$ are deformation measures, $\boldsymbol{\alpha}$ is a generalized spin velocity, and the quantities $A_{\alpha\beta}$, $B_{\alpha\beta}$ are measures of the inertia properties of the rigid director triad. The notation $(\hat{\ })$, $(\check{\ })$ indicates arc differentiation with respect to the undeformed and deformed arc lengths $S$ and $s$, respectively. The superposed dot denotes the material time derivative holding $S$ fixed. Finally, $\lambda = ds/dS$ is the stretch, $\mathbf{t}$ is the unit tangent to $c$ and $\mathbf{r}$ is the position vector of points on $c$. Greek subscripts take the values 1, 2, 3 and refer to anholonomic components of tensors with respect to the directors $\mathbf{d}_\alpha$, e.g. $\tau_\alpha = \boldsymbol{\tau} . \mathbf{d}_\alpha$. The usual summation convention applies.

To describe the plane motions of a Cosserat curve, we choose a rectangular cartesian coordinate system $x_i$ and require that the curve $c$ and two of its directors $\mathbf{d}_1$, $\mathbf{d}_2$ lie in the $x_1$–$x_2$ plane. Since the directors $\mathbf{d}_\alpha$ form an orthogonal triad, $\mathbf{d}_3$ always remains aligned with the $x_3$-axis. Except for the coordinate axes $x_i$, Arabic numeral subscripts will represent anholonomic components, exclusively. Since the directors have a single rotational degree of freedom, we can define their orientation by an angle $\varphi$ which $\mathbf{d}_1$ makes with the $x_1$-axis. Similarly, the orientation of the unit tangent $\mathbf{t}$ can be specified by an angle $\theta$ which $\mathbf{t}$ makes with the $x_1$-axis. Hence, we have

$$
\begin{aligned}
\mathbf{t} &= (\cos\theta, \sin\theta, 0) \\
\mathbf{d}_1 &= (\cos\varphi, \sin\varphi, 0) \\
\mathbf{d}_2 &= (-\sin\varphi, \cos\varphi, 0) \\
\mathbf{d}_3 &= (0, 0, 1).
\end{aligned} \tag{2.5}
$$

From these representations we can show that the non-vanishing components of $t_\alpha$ and $w_{\alpha\beta}$ are

$$
\begin{aligned}
t_1 &= \cos(\theta - \varphi), \qquad & t_2 &= \sin(\theta - \varphi) \\
w_{12} &= -\check{\varphi}, \qquad & w_{21} &= \check{\varphi}.
\end{aligned} \tag{2.6}
$$

Substituting equations (2.6) into (2.3) and (2.2), the non-zero deformation measures and constitutive relations are:

$$y_1 = \lambda t_1 = \lambda\cos(\theta-\varphi), \qquad y_2 = \lambda t_2 = \lambda\sin(\theta-\varphi)$$
$$F_{21} = \hat{\varphi} = -F_{12} \tag{2.7}$$

$$\tau_1 = \rho\lambda\frac{\partial\varepsilon}{\partial y_1}, \qquad \tau_2 = \rho\lambda\frac{\partial\varepsilon}{\partial y_2}, \qquad m_3 = \rho\lambda\frac{\partial\varepsilon}{\partial\hat{\varphi}}, \tag{2.8}$$

Finally, the anholonomic components of the spin velocity $\mathbf{\Omega}$ are

$$\Omega_1 = \Omega_2 = 0, \qquad \Omega_3 = \dot{\varphi}.$$

The preceding results imply the equations of motion become

$$\frac{\partial\tau_1}{\partial s} - \check{\varphi}\tau_2 + \rho f_1 = \rho\dot{v}_1 - \rho\dot{\varphi}v_2$$
$$\frac{\partial\tau_2}{\partial s} + \check{\varphi}\tau_1 + \rho f_2 = \rho\dot{v}_2 + \rho\dot{\varphi}v_1 \tag{2.9}$$

$$f_3 = 0$$
$$l_1 = \dot{\alpha}_1 - \dot{\varphi}\alpha_2$$
$$l_2 = \dot{\alpha}_2 + \dot{\varphi}\alpha_1 \tag{2.10}$$

$$\frac{\partial m_3}{\partial s} - (\tau_1 t_2 - \tau_2 t_1) + \rho l_3 = \rho\dot{\alpha}_3 \tag{2.11}$$

where

$$\alpha_1 = B_{13}\dot{\varphi}, \qquad \alpha_2 = B_{23}\dot{\varphi}, \qquad \alpha_3 = B_{33}\dot{\varphi}.$$

Noting that the body couples $l_1$, $l_2$ tend to deform the curve out of the plane of deformation, we assume these couples vanish. Equations (2.10) then imply

$$(B_{13}^2 + B_{23}^2)\dot{\varphi}^2 = 0.$$

This equation is satisfied for arbitrary $\dot{\varphi}$ provided $B_{13} = B_{23} = 0$. This restriction on the matrix $B_{\alpha\beta}$ can be viewed as analogous to the requirement in classical beam and rod theory that a plane deformation can occur only in a principal plane of the cross section.

At this point it is convenient to define a set of strains $z_\alpha$, $\varkappa_{\alpha\beta}$ which vanish in the reference configuration:

$$z_\alpha = y_\alpha - \mathbf{D}_\alpha \,.\, \hat{\mathbf{R}}$$
$$\varkappa_{\alpha\beta} = F_{\alpha\beta} - \mathbf{D}_\alpha \,.\, \hat{\mathbf{D}}_\beta \tag{2.12}$$

where $\mathbf{R}$ and $\mathbf{D}_\alpha$ are the position vector and directors, respectively, in the reference configuration. We can evaluate the terms $\mathbf{D}_\alpha \,.\, \hat{\mathbf{R}}$ and $\mathbf{D}_\alpha \,.\, \hat{\mathbf{D}}_\beta$ above by defining angles $\Theta$, $\Phi$ such that

$$\Theta(S) = \theta|_{t=0}, \qquad \Phi(S) = \varphi|_{t=0}.$$

These definitions lead to the expressions:

$$\begin{aligned} \mathbf{D}_1 \,.\, \hat{\mathbf{R}} &= \cos(\Theta - \Phi), \qquad \mathbf{D}_2 \,.\, \hat{\mathbf{R}} = \sin(\Theta - \Phi) \\ \mathbf{D}_2 \,.\, \hat{\mathbf{D}}_1 &= \hat{\Phi} = -\mathbf{D}_1 \,.\, \hat{\mathbf{D}}_2. \end{aligned} \tag{2.13}$$

Without loss of generality we can take $\mathbf{D}_1$ aligned with the tangent vector $\hat{\mathbf{R}}$ by choosing

$$\Phi = \Theta, \qquad \hat{\Phi} = \hat{\Theta} = K \tag{2.14}$$

where $K$ is the curvature of the undeformed curve. The relevant strains then become

$$z_1 = y_1 - 1, \qquad z_2 = y_2, \qquad \varkappa = \hat{\varphi} - K \tag{2.15}$$

where we have defined $\varkappa = \varkappa_{21} = -\varkappa_{12}$. Hence, the general equations for plane motion of Cosserat curves become

$$\begin{aligned} &\frac{\partial \tau_1}{\partial s} - \check{\varphi}\tau_2 + \rho f_1 = \rho \dot{v}_1 - \rho \dot{\varphi} v_2 \\ &\frac{\partial \tau_2}{\partial s} + \check{\varphi}\tau_1 + \rho f_2 = \rho \dot{v}_2 + \rho \dot{\varphi} v_1 \\ &\frac{\partial m}{\partial s} - (\tau_1 t_2 - \tau_2 t_1) + \rho l = \rho B \ddot{\varphi} \end{aligned} \tag{2.16}$$

$$\tau_1 = \rho\lambda \frac{\partial \varepsilon}{\partial z_1}, \qquad \tau_2 = \rho\lambda \frac{\partial \varepsilon}{\partial z_2}, \qquad m = \rho\lambda \frac{\partial \varepsilon}{\partial \varkappa} \tag{2.17}$$

$$\begin{aligned} z_1 &= \lambda t_1 - 1, \qquad z_2 = \lambda t_2, \qquad \varkappa = \hat{\varphi} - K \\ t_1 &= \cos(\theta - \varphi), \qquad t_2 = \sin(\theta - \varphi). \end{aligned} \tag{2.18}$$

The couple stress equation $(2.16)_3$ was obtained from (2.11) by dropping the subscript 3 from the appropriate quantities. To obtain a determinate theory in the sense that the number of equations equal the number of unknowns, we must add the following equations to the set (2.16) to (2.18):

$$\begin{aligned} &\frac{\partial x_1}{\partial s} = \cos\theta, \qquad \frac{\partial x_2}{\partial s} = \sin\theta \\ &\dot{x}_1 = v_1 \cos\varphi - v_2 \sin\varphi \\ &\dot{x}_2 = v_1 \sin\varphi + v_2 \cos\varphi. \end{aligned}$$

We note in passing that the above theory includes a nonlinear, classical type elastica theory as a special case by requiring the director frame to rotate with the tangent vector $\mathbf{t}$. For this special case we have

$$\varphi = \theta, \qquad t_1 = 1, \qquad t_2 = 0.$$

The subscripts 1, 2 now refer to components along the tangent and normal to the curve, respectively. The strains become

$$z_1 = \lambda - 1 \equiv \delta, \qquad z_2 = 0, \qquad \varkappa = \hat{\theta} - K \equiv \mu$$

where $\delta$, $\mu$ are the classical extension and bending measures, respectively. Hence, the constitutive equations (2.17) become

$$\tau_1 = \rho\lambda\frac{\partial\varepsilon}{\partial\delta}, \qquad m = \rho\lambda\frac{\partial\varepsilon}{\partial\mu}.$$

The above form of the constitutive relations for an elastica were obtained by Antman [5] from the three-dimensional (nonlinear) constitutive equations of elasticity. We observe that the shear stress $\tau_2$ is indeterminate with respect to constitutive equations, but can be determined from the equations of motion, which become

$$\begin{aligned}\frac{\partial\tau_1}{\partial s}-\check{\theta}\tau_2+\rho f_1 &= \rho\dot{v}_1-\rho\dot{\theta}v_2\\ \frac{\partial\tau_2}{\partial s}+\check{\theta}\tau_1+\rho f_2 &= \rho\dot{v}_2+\rho\dot{\theta}v_1\\ \frac{\partial m}{\partial s}+\tau_2+\rho l &= \rho B\ddot{\theta}.\end{aligned} \tag{2.19}$$

If the inertia terms are omitted from equations (2.19), we recover the equilibrium equations of the plane elastica (cf. Love [6]). Hence, we see that the model of a Cosserat curve yields a nonlinear theory of the plane elastica as a special case.

In order to develop the linear theory of a plane Cosserat curve, we define a displacement vector $\mathbf{u}$ and a set of director displacements $\boldsymbol{\theta}_\alpha$ by the equations

$$\mathbf{r} = \mathbf{R}+\mathbf{u}, \qquad \mathbf{d}_\alpha = \mathbf{D}_\alpha+\boldsymbol{\theta}_\alpha. \tag{2.20}$$

We consider $\mathbf{u}$, $\boldsymbol{\theta}_\alpha$ and their first derivatives with respect to arc length or time to be infinitesimal such that all second or higher order terms in these quantities can be neglected in the general equations. Equations (2.3), (2.12) and (2.20) then imply that the linearized strains are

$$\begin{aligned} z_\alpha &\cong \boldsymbol{\theta}_\alpha \cdot \hat{\mathbf{R}}+\mathbf{D}_\alpha \cdot \hat{\mathbf{u}}\\ \varkappa_{\alpha\beta} &\cong \boldsymbol{\theta}_\alpha \cdot \hat{\mathbf{D}}_\beta+\mathbf{D}_\alpha \cdot \hat{\boldsymbol{\theta}}_\beta.\end{aligned} \tag{2.21}$$

Since the directors are assumed to undergo an infinitesimal rotation, we can define a small angle $\psi = \varphi-\Theta$ such that the director displacements are given by

$$\boldsymbol{\theta}_1 = \psi\mathbf{D}_2, \qquad \boldsymbol{\theta}_2 = -\psi\mathbf{D}_1, \qquad \boldsymbol{\theta}_3 = 0. \tag{2.22}$$

Also, within the linear theory the displacement vector $\mathbf{u}$ can be expressed as

$$\mathbf{u} \cong u_1\mathbf{D}_1+u_2\mathbf{D}_2. \tag{2.23}$$

Equations (2.22) and (2.23) imply the strains become

$$z_1 = \frac{\partial u_1}{\partial S}-Ku_2, \qquad z_2 = \frac{\partial u_2}{\partial S}-\psi+Ku_1, \qquad \varkappa = \hat{\psi} = \hat{\varphi}-K. \tag{2.24}$$

The rotation of the tangent vector $\mathbf{t}$ can be defined in terms of a small angle $\chi$ such that

$$\mathbf{t} \cong \mathbf{D}_1+\chi\mathbf{D}_2.$$

This representation of $\mathbf{t}$ leads to the following expressions for the stretch $\lambda$ and the angle $\chi$:

$$\lambda \cong 1+\frac{\partial u_1}{\partial S}-Ku_2, \qquad \chi = \frac{\partial u_2}{\partial S}+Ku_1. \tag{2.25}$$

Comparing (2.24) and (2.25), the strains $z_1$, $z_2$ can be written as

$$z_1 = \lambda-1, \qquad z_2 = \chi-\psi.$$

For the purpose of developing linear constitutive relations we assume the strain energy function $\varepsilon$ is given by

$$\varepsilon = \tfrac{1}{2}c_{\alpha\beta}\eta_\alpha\eta_\beta \tag{2.26}$$

where $c_{\alpha\beta}$ is a constant, symmetric matrix of material parameters and $\eta_\alpha$ is the set of strains $(z_1, z_2, \varkappa)$. Equations (2.17) and (2.26), together with the conservation of mass $\rho\lambda = \rho_0$, yield the constitutive equations

$$\tau_1 = \rho_0 c_{1\beta}\eta_\beta, \qquad \tau_2 = \rho_0 c_{2\beta}\eta_\beta, \qquad m = \rho_0 c_{3\beta}\eta_\beta. \tag{2.27}$$

Using the preceding results in equations (2.16) and assuming constant initial density, we can show that the linear displacement equations of motion are

$$\begin{aligned}
&c_{11}\frac{\partial}{\partial S}\left(\frac{\partial u_1}{\partial S}-Ku_2\right)+c_{12}\frac{\partial}{\partial S}\left(\frac{\partial u_2}{\partial S}-\psi+Ku_1\right)+c_{13}\frac{\partial^2\psi}{\partial S^2}\\
&\quad -K\left[c_{22}\left(\frac{\partial u_2}{\partial S}-\psi+Ku_1\right)+c_{12}\left(\frac{\partial u_1}{\partial S}-Ku_2\right)+c_{23}\frac{\partial\psi}{\partial S}\right]+f_1 = \frac{\partial^2 u_1}{\partial t^2}\\
&c_{22}\frac{\partial}{\partial S}\left(\frac{\partial u_2}{\partial S}-\psi+Ku_1\right)+c_{12}\frac{\partial}{\partial S}\left(\frac{\partial u_1}{\partial S}-Ku_2\right)+c_{23}\frac{\partial^2\psi}{\partial S^2}\\
&\quad +K\left[c_{11}\left(\frac{\partial u_1}{\partial S}-Ku_2\right)+c_{12}\left(\frac{\partial u_2}{\partial S}-\psi+Ku_1\right)+c_{13}\frac{\partial\psi}{\partial S}\right]+f_2 = \frac{\partial^2 u_2}{\partial t^2}\\
&c_{33}\frac{\partial^2\psi}{\partial S^2}+c_{13}\frac{\partial}{\partial S}\left(\frac{\partial u_1}{\partial S}-Ku_2\right)+c_{23}\frac{\partial}{\partial S}\left(\frac{\partial u_2}{\partial S}-\psi+Ku_1\right)+c_{22}\left(\frac{\partial u_2}{\partial S}-\psi+Ku_1\right)\\
&\quad +c_{12}\left(\frac{\partial u_1}{\partial S}-Ku_2\right)+c_{23}\frac{\partial\psi}{\partial S}+l = B\frac{\partial^2\psi}{\partial t^2}.
\end{aligned} \tag{2.28}$$

Recalling (2.23) and the fact that the directors $\mathbf{D}_1$, $\mathbf{D}_2$ were chosen along the tangent and normal to the undeformed curve, the displacements $u_1$, $u_2$ represent extensional and transverse displacements of the curve. Thus, equations (2.28) imply that extension, flexure and rotation of the directors are interacting effects. This statement remains valid even when the curve is initially straight, i.e. when $K(S) = 0$.

As a special case, we assume that the matrix $c_{\alpha\beta}$ is diagonal with elements $c_{11} = a$, $c_{22} = b$, $c_{33} = c$. Then for an initially straight curve, equations (2.28) reduce to

$$\begin{aligned}
a\,\frac{\partial^2 u_1}{\partial S^2}+f_1 &= \frac{\partial^2 u_1}{\partial t^2}\\
b\,\frac{\partial}{\partial S}\left(\frac{\partial u_2}{\partial S}-\psi\right)+f_2 &= \frac{\partial^2 u_2}{\partial t^2}\\
c\,\frac{\partial^2\psi}{\partial S^2}+b\left(\frac{\partial u_2}{\partial S}-\psi\right)+l &= B\frac{\partial^2\psi}{\partial t^2}.
\end{aligned} \tag{2.29}$$

This special theory implies a partial decoupling of effects, i.e. extensional motion is governed by a one-dimensional wave equation. However, flexure of the curve and rotation of the director frame remain coupled. The rotational angle $\psi$ can be eliminated from equations $(2.29)_{2,3}$ yielding a single equation for the transverse displacement $u_2$. Hence, for zero body forces we find

$$c\frac{\partial^4 u_2}{\partial S^4} - \left(B + \frac{c}{b}\right)\frac{\partial^4 u_2}{\partial S^2 \partial t^2} + \frac{\partial^2 u_2}{\partial t^2} + \frac{B}{b}\frac{\partial^4 u_2}{\partial t^4} = 0. \tag{2.30}$$

This equation has the same form as the classical equation for transverse displacement of Timoshenko beam theory, which takes into account shear deformation and rotatory inertia. An equation equivalent to (2.30) was also obtained by Green, Laws and Naghdi [7] from their linear theory of straight elastic rods, based on a two director model of a rod.

## 3. STABILITY OF GENERAL MOTIONS OF DIRECTED CURVES

Within the context of stability analysis, we distinguish between two motions of a directed curve. The first is the undisturbed motion, i.e. that motion which is assumed known and the stability of which is to be investigated. The second is any neighboring motion called the disturbed or perturbed motion. We introduce the concept of a metric functional, i.e. a non-negative measure of "distance" between the two motions. As used here, the term distance has a very general connotation and may refer to velocity change, temperature rise, stress increase or any other quantities which serve to distinguish the change in the undisturbed motion. Assuming the undisturbed motion of the directed curve undergoes a disturbance at some time $t_0$, we define a metric $M_0$ as a measure of this disturbance.† A metric $M(t)$ is then defined as a measure of the disturbance at any time $t > t_0$. We assume the metrics $M_0$, $M(t)$ are defined such that they vanish only when evaluated along the undisturbed motion of the curve. The undisturbed motion of the directed curve is then said to be *stable* with respect to the metrics $M_0$, $M(t)$ provided the following conditions are satisfied:

(a) $M(t)$ is a continuous function of $t$.
(b) $M(t)$ is continuous with respect to $M_0$ at $t = t_0$, i.e. given any $\epsilon_1 > 0$, there exists a $\delta_1(\epsilon_1, t_0) > 0$ such that at the initial time $t_0$, $M_0 < \delta_1$ implies $M(t_0) < \epsilon_1$.
(c) Given any $\epsilon_2 > 0$, there exists a $\delta_2(\epsilon_2, t_0) > 0$ such that $M_0 < \delta_2$ implies $M(t) < \epsilon_2$ for $t > t_0$.

This definition and the theorem which follows are due to Movchan [4].

THEOREM. The undisturbed motion is stable with respect to the metrics $M_0$, $M(t)$ if and only if there exists in the neighborhood $M(t) < R$, $R > 0$ of the undisturbed motion a functional $V(t)$ such that

(i) $V(t) \geq 0$.
(ii) $V(t)$ is a non-increasing function of $t$.
(iii) Given any $\epsilon_1 > 0$, there exists a $\delta_1(\epsilon_1, t_0) > 0$ such that $M_0 < \delta_1$ implies $V(t_0) < \epsilon_1$.
(iv) Given any $\epsilon_2 > 0$, there exists a $\delta_2(\epsilon_2) > 0$ such that $M(t) \geq \delta_2$ implies $V(t) \geq \epsilon_2$.

Movchan also proved that the stability of the undisturbed motion according to the definition given implies its uniqueness in the sense that if $M_0 = 0$ at $t = t_0$, then $M(t)$ must vanish for all $t > t_0$.

†We suppress the functional dependence of the metrics and other functionals on the quantities which characterize the distance between the two motions.

We now consider an appropriate functional $V(t)$ for general motions of elastic directed curves. The integral form of energy conservation is (see Ref. [2])

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_c \tfrac{1}{2}[\rho\mathbf{v}\cdot\mathbf{v}+\tfrac{1}{2}A^{\alpha\beta}\mathbf{w}_\alpha\cdot\mathbf{w}_\beta+\rho\varepsilon]\,\mathrm{d}s = \int_c \rho(\mathbf{f}\cdot\mathbf{v}+\mathbf{h}^\alpha\cdot\mathbf{w}_\alpha)\,\mathrm{d}s+(\boldsymbol{\tau}\cdot\mathbf{v}+\boldsymbol{\mu}^\alpha\cdot\mathbf{w}_\alpha)\Big|_{s_1}^{s_2} \tag{3.1}$$

where $\mathbf{w}_\alpha = \mathbf{d}_\alpha$ and $\mathbf{h}^\alpha$, $\boldsymbol{\mu}^\alpha$ are the body force and the stress vectors associated with the directors. We assume the curve $c$ undergoes a disturbance at time $t_0$ and that for $t > t_0$ there exists a perturbed motion defined by the functions

$$\mathbf{r}^* = \mathbf{r}^*(s^*, t), \qquad \mathbf{d}_\alpha^* = \mathbf{d}_\alpha^*(s^*, t) \tag{3.2}$$

taking $c$ into $c^*$ such that

$$s^* = s^*(s, t) \tag{3.3}$$

where $s^*$ is the arc length of the curve in the configuration $c^*$. Applying the conservation of energy (3.1) to the perturbed motion, as well as the conservation of mass and the mapping (3.3), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_c (\tfrac{1}{2}\mathbf{v}\cdot\mathbf{v}+\tfrac{1}{2}A^{\alpha\beta}\mathbf{w}_\alpha\cdot\mathbf{w}_\beta+\varepsilon)^*\rho\,\mathrm{d}s = \int_c (\mathbf{f}\cdot\mathbf{v}+\mathbf{h}^\alpha\cdot\mathbf{w}_\alpha)^*\rho\,\mathrm{d}s+(\boldsymbol{\tau}\cdot\mathbf{v}+\boldsymbol{\mu}^\alpha\cdot\mathbf{w}_\alpha)^*\Big|_{s_1}^{s_2} \tag{3.4}$$

where the notation ( )* indicates the enclosed functions are defined along the perturbed configuration $c^*$. Subtraction of (3.1) from (3.4) yields the result

$$\frac{\mathrm{d}}{\mathrm{d}t}K(t)-H(t) = 0 \tag{3.5}$$

where $K(t)$ and $H(t)$ are defined by

$$K(t) = \int_c [\tfrac{1}{2}(\mathbf{v}^*\cdot\mathbf{v}^*-\mathbf{v}\cdot\mathbf{v})+\tfrac{1}{2}A^{\alpha\beta}(\mathbf{w}_\alpha^*\cdot\mathbf{w}_\beta^*-\mathbf{w}_\alpha\cdot\mathbf{w}_\beta)+(\varepsilon^*-\varepsilon)]\rho\,\mathrm{d}s \tag{3.6}$$

$$H(t) = \int_c [(\mathbf{f}^*\cdot\mathbf{v}^*-\mathbf{f}\cdot\mathbf{v})+(\mathbf{h}^{*\alpha}\cdot\mathbf{w}_\alpha^*-\mathbf{h}^\alpha\cdot\mathbf{w}_\alpha)]\rho\,\mathrm{d}s+[(\boldsymbol{\tau}^*\cdot\mathbf{v}^*-\boldsymbol{\tau}\cdot\mathbf{v})+(\boldsymbol{\mu}^{*\alpha}\cdot\mathbf{w}_\alpha^*-\boldsymbol{\mu}^\alpha\cdot\mathbf{w}_\alpha)]\Big|_{s_1}^{s_2}. \tag{3.7}$$

We now define a Movchan–Liapounov functional $V(t)$ to be

$$V(t) = K(t)-\int H(t)\,\mathrm{d}t. \tag{3.8}$$

By virtue of the differential equation (3.5) we see that $V(t)$ is constant. Hence this functional satisfies condition (ii) of Movchan's stability theorem and is at least a candidate functional for stability analysis.

It is often convenient to introduce displacement functions $\mathbf{u}, \boldsymbol{\theta}_\alpha$† according to the equations

$$\mathbf{r}^* = \mathbf{r}+\mathbf{u}, \qquad \mathbf{d}_\alpha^* = \mathbf{d}_\alpha+\boldsymbol{\theta}_\alpha. \tag{3.9}$$

† Note that these displacements in general have a different meaning than those introduced by (2.20).

Using equations (3.9), the quantities $K(t)$ and $H(t)$ become

$$K(t) = \int_c [\tfrac{1}{2}(2\dot{\mathbf{u}}\,.\,\mathbf{v}+\dot{\mathbf{u}}\,.\,\dot{\mathbf{u}})+\tfrac{1}{2}A^{\alpha\beta}(2\mathbf{w}_\alpha\,.\,\dot{\boldsymbol{\theta}}_\beta+\dot{\boldsymbol{\theta}}_\alpha\,.\,\dot{\boldsymbol{\theta}}_\beta)+(\varepsilon^*-\varepsilon)]\rho\,\mathrm{d}s \tag{3.10}$$

$$\begin{aligned} H(t) = \int_c [(\mathbf{f}^*-\mathbf{f})\,.\,\mathbf{v}+\mathbf{f}^*\,.\,\dot{\mathbf{u}}+(\mathbf{h}^{*\alpha}-\mathbf{h}^\alpha)\,.\,\mathbf{w}_\alpha+\mathbf{h}^{*\alpha}\,.\,\dot{\boldsymbol{\theta}}_\alpha]\rho\,\mathrm{d}s \\ +[(\boldsymbol{\tau}^*-\boldsymbol{\tau})\,.\,\mathbf{v}+\boldsymbol{\tau}^*\,.\,\dot{\mathbf{u}}+(\boldsymbol{\mu}^{*\alpha}-\boldsymbol{\mu}^\alpha)\,.\,\mathbf{w}_\alpha+\boldsymbol{\mu}^{*\alpha}\,.\,\dot{\boldsymbol{\theta}}_\alpha]\Big|_{s_1}^{s_1}. \end{aligned} \tag{3.11}$$

Hence, equations (3.10) and (3.11) express $V(t)$ in terms of displacements and velocities from the undisturbed motion of the directed curve. We note that no assumptions have been made up to this point regarding the loading functions.

The special case of dead loading yields a simplification in the functional $H(t)$. By dead loading we mean that the body forces and end forces in the disturbed motion remain unchanged from their values in the undisturbed motion. Thus, for dead loading

$$\mathbf{f}^* = \mathbf{f}, \qquad \mathbf{h}^{*\alpha} = \mathbf{h}^\alpha \quad \text{along} \quad c$$

$$\boldsymbol{\tau}^* = \boldsymbol{\tau}, \qquad \boldsymbol{\mu}^{*\alpha} = \boldsymbol{\mu}^\alpha \quad \text{at} \quad s = s_1, s_2$$

and $H(t)$ becomes

$$H(t) = \int_c (\mathbf{f}\,.\,\dot{\mathbf{u}}+\mathbf{h}^\alpha\,.\,\dot{\boldsymbol{\theta}}_\alpha)\rho\,\mathrm{d}s+(\boldsymbol{\tau}\,.\,\dot{\mathbf{u}}+\boldsymbol{\mu}^\alpha\,.\,\dot{\boldsymbol{\theta}}_\alpha)\Big|_{s_1}^{s_2}.$$

If in addition to the assumption of dead loading, the undisturbed motion is restricted to be an equilibrium configuration, we can show that $V(t)$ takes the form

$$V(t) = \int_c [\tfrac{1}{2}\dot{\mathbf{u}}\,.\,\dot{\mathbf{u}}+\tfrac{1}{2}A^{\alpha\beta}\dot{\boldsymbol{\theta}}_\alpha\,.\,\dot{\boldsymbol{\theta}}_\beta+(\varepsilon^*-\varepsilon)]\rho\,\mathrm{d}s-\int_c (\mathbf{f}\,.\,\mathbf{u}+\mathbf{h}^\alpha\,.\,\boldsymbol{\theta}_\alpha)\rho\,\mathrm{d}s-(\boldsymbol{\tau}\,.\,\mathbf{u}+\boldsymbol{\mu}^\alpha\,.\,\boldsymbol{\theta}_\alpha)\Big|_{s_1}^{s_2}. \tag{3.12}$$

Finally, when the equilibrium configuration is the natural or unloaded state, the requirement that $V(t)$ be non-negative reduces to

$$\int_c \rho\varepsilon^*\,\mathrm{d}s \geq 0.$$

This condition imposes a restriction which the strain energy function must satisfy for the stability of the disturbed motion. An analogous condition for three-dimensional elasticity was presented by Knops and Wilkes [8].

## 4. STABILITY OF A PLANE, SIMPLY-SUPPORTED COSSERAT CURVE

In this application of Movchan's theorem, we restrict ourselves to the case of linear motions from the undeformed state of the curve. It can then be shown that the general functional $V(t)$ defined by equations (3.8), (3.10) and (3.11) can be reduced to the form:

$$V(t) = \int_0^L (\tfrac{1}{2}\dot{u}_\alpha\dot{u}_\alpha+\tfrac{1}{2}B\dot{\psi}^2+\varepsilon)\rho_0\,\mathrm{d}S-\int \mathrm{d}t\left[\int_0^L (f_\alpha\dot{u}_\alpha+l\dot{\psi})\rho_0\,\mathrm{d}S+(\tau_\alpha\dot{u}_\alpha+m\dot{\psi})\Big|_0^L\right] \tag{4.1}$$

where $\varepsilon$ is given by (2.26) and where $L$ is the length of the curve in its natural state. It can easily be verified that the time rate of $V(t)$ above vanishes by virtue of the equations of motion (2.28). For simplicity we assume that the body forces $f_\alpha$, $l$ vanish and that the undeformed state is perturbed by imparting a set of displacements and velocities to the system at time $t = 0$. Moreover, we treat the case of zero initial curvature $K$. Hence, the governing equations become

$$\begin{aligned} &c_{11}\frac{\partial^2 u_1}{\partial S^2}+c_{12}\frac{\partial}{\partial S}\left(\frac{\partial u_2}{\partial S}-\psi\right)+c_{13}\frac{\partial^2\psi}{\partial S^2} = \frac{\partial^2 u_1}{\partial t^2} \\ &c_{22}\frac{\partial}{\partial S}\left(\frac{\partial u_2}{\partial S}-\psi\right)+c_{12}\frac{\partial^2 u_1}{\partial S^2}+c_{23}\frac{\partial^2\psi}{\partial S^2} = \frac{\partial^2 u_2}{\partial t^2} \\ &c_{33}\frac{\partial^2\psi}{\partial S^2}+c_{13}\frac{\partial^2 u_1}{\partial S^2}+c_{23}\frac{\partial}{\partial S}\left(\frac{\partial u_2}{\partial S}-\psi\right) \\ &\quad +c_{22}\left(\frac{\partial u_2}{\partial S}-\psi\right)+c_{12}\frac{\partial u_1}{\partial S}+c_{23}\frac{\partial\psi}{\partial S} = B\frac{\partial^2\psi}{\partial t^2}. \end{aligned} \tag{4.2}$$

A simply-supported Cosserat curve is defined by the boundary conditions

$$\begin{aligned} u_1(0,t) &= u_2(0,t) = m(0,t) = 0 \\ u_1(L,t) &= u_2(L,t) = m(L,t) = 0. \end{aligned} \tag{4.3}$$

The functional $V(t)$ takes the form

$$V(t) = \tfrac{1}{2}\int_0^L (\dot{u}_\alpha\dot{u}_\alpha + B\dot{\psi}^2 + c_{\alpha\beta}\eta_\alpha\eta_\beta)\rho_0\,\mathrm{d}S \tag{4.4}$$

where

$$\eta_1 = z_1 = \frac{\partial u_1}{\partial S}, \qquad \eta_2 = z_2 = \frac{\partial u_2}{\partial S}-\psi, \qquad \eta_3 = \varkappa = \frac{\partial\psi}{\partial S}.$$

To apply Movchan's theorem, we choose the following metric functionals

$$M_0 \equiv V(0), \qquad M(t) = \int_0^L (u_\alpha u_\alpha + \psi^2)\,\mathrm{d}S. \tag{4.5}$$

We assume that $u_\alpha$ and $\psi$ are continuous functions of $t$ so that $M(t)$ is continuous. The definition $(4.5)_1$ trivially satisfies condition (iii) of Movchan's theorem. We now make the assumption that the strain energy function is a positive definite quadratic form, i.e. there exists a positive constant $c$ such that

$$c_{\alpha\beta}\eta_\alpha\eta_\beta \geq c\eta_\alpha\eta_\alpha. \tag{4.6}$$

Hence, noting that the inertia coefficient is positive, we obtain the estimate

$$V(t) \geq \tfrac{1}{2}\rho_0 c\int_0^L \left[\left(\frac{\partial u_1}{\partial S}\right)^2 + \left(\frac{\partial u_2}{\partial S}-\psi\right)^2 + \left(\frac{\partial\psi}{\partial S}\right)^2\right]\mathrm{d}S. \tag{4.7}$$

Performing the change of variables

$$S = L\xi, \qquad U_\alpha(\xi,t) = \frac{1}{L}u_\alpha(L\xi,t) \tag{4.8}$$

inequality (4.7) becomes

$$V(t) \geq \tfrac{1}{2}\rho_0 c \min\left(1, \frac{1}{L^2}\right) \int_0^1 \left[\left(\frac{\partial U_1}{\partial \xi}\right)^2 + \left(\frac{\partial U_2}{\partial \xi} - \psi\right)^2 + \left(\frac{\partial \psi}{\partial \xi}\right)^2\right] d\xi. \tag{4.9}$$

Expanding the second term in the above integrand, integrating by parts and rearranging, we can show that (4.9) takes the form

$$V(t) \geq \tfrac{1}{2}\rho_0 c \min\left(1, \frac{1}{L^2}\right) \int_0^1 \left[\left(\frac{\partial U_1}{\partial \xi}\right)^2 + \left(\frac{\partial U_2}{\partial \xi}\right)^2 + \psi^2 - U_2^2\right] d\xi. \tag{4.10}$$

By an application of Schwarz's inequality it follows easily that

$$\int_0^1 \left(\frac{\partial U_\alpha}{\partial \xi}\right)^2 d\xi \geq 2 \int_0^1 U_\alpha^2(\xi, t)\, d\xi. \tag{4.11}$$

Hence, applying (4.11) to (4.10) and returning to the original variables, we obtain

$$V(t) \geq C_1 M(t) \tag{4.12}$$

where

$$C_1 = \tfrac{1}{2}\rho_0 c \left[\min\left(1, \frac{1}{L^2}\right)\right]^2 > 0. \tag{4.13}$$

Inequality (4.12) implies that conditions (i) and (iv) are satisfied. Since by definition $V(t)$ is constant, inequality (4.12) and the definition of $M_0$ imply that

$$M(t) \leq \frac{1}{C_1} V(t) = \frac{1}{C_1} V(0) = \frac{1}{C_1} M_0. \tag{4.14}$$

Hence, we have shown that the undeformed state of a simply-supported Cosserat curve is stable, provided the strain energy function is a positive definite quadratic form. Moreover, the solution to equations (4.2) under arbitrary initial conditions is unique. We note that (4.5) and (4.14) imply that the displacements $u_\alpha, \psi$ are small in the average sense of $M(t)$ for all $t > 0$ when the initial energy imparted to the system is small.

A stability analysis can also be given in this example using the metric

$$\overline{M}(t) = \int_0^L (u_\alpha u_\alpha + \psi^2 + \dot{u}_\alpha \dot{u}_\alpha + \dot{\psi}^2)\, dS$$

where $u_\alpha, \psi$ are assumed to be continuously differentiable functions of time. Then corresponding to inequality (4.12) we can show that

$$V(t) \geq \overline{C}_1 \overline{M}(t)$$

where $\overline{C}_1$ is given by (4.13) with $c$ replaced by min $(1, B, c)$. A positive definite strain energy function is again a sufficient condition for the stability of the undeformed state of the curve. Stability with respect to $\overline{M}(t)$ implies the stronger result that the velocities, in addition to the displacements remain small in the average sense of $\overline{M}(t)$ for all $t > 0$.

Our result obviously remains valid for the special case when $c_{\alpha\beta}$ is a diagonal matrix, i.e. when the governing equations are given by (2.29) with zero body forces. This case is also included in the work of Green, Knops and Laws [9], who investigated the stability of an initially straight rod subjected to a simple extension.

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**Абстракт**—Определяется нелинейная динамическая теория плоских движений класса кривых Коссера. В качестве специального случая, она заключает классический тип теории упругости. Доказывается, что недерформированное состояние свободно опертой кривой является устойчивое по отношению к соответствующим метрикам, если только функция знергии деформации положителвная.